

Report

A project on “A Study on Matrices & Vector Spaces of Graphs” was undertaken by the students of Department of Mathematics under the guidance of Sri Arabinda Pandab, HOD Mathematics. It took two months (Feb & March 2020) to carry out the project. Graphs are the quintessential objects of study in discrete mathematics. They are usually described as a set of vertices, V , that are connected by a set of edges, E , each of which is a pair of vertices. Graphs encode connections and are one of the most commonly used representations of data. Mathematicians often define graphs abstractly. For example, we define a path graph to be a graph with vertex set $V = \{1, \dots, n\}$ and $E = \{(i, i+1) : 1 \leq i < n\}$. Or a number theorist might consider a graph with $V = \{1, \dots, n\}$ and E the set of pairs (i, j) for which i divides j . The public is more familiar with social network graphs, in which each person is a vertex and edges exist between pairs of people who are “friends”.

Since a graph is completely determined by specifying either its adjacency structure or its incidence structure, these specifications provide far more efficient ways of representing a large or complicated graph than a pictorial representation. As computers are more adept at manipulating numbers than at recognising pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. In this chapter, students studied various types of matrices associated with a graph.

Finally, the project was completed and submitted on 20th March 2020.

Arabinda Pandab
HOD, Mathematics

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Chapter 1

INTRODUCTION AND SUMMARY

1.1 Introduction

Graphs are the quintessential objects of study in discrete mathematics. They are usually described as a set of vertices, V , that are connected by a set of edges, E , each of which is a pair of vertices. Graphs encode connections and are one of the most commonly used representations of data. Mathematicians often define graphs abstractly. For example, we define a path graph to be a graph with vertex set $V = \{1, \dots, n\}$ and $E = \{(i, i+1) : 1 \leq i < n\}$. Or a number theorist might consider a graph with $V = \{1, \dots, n\}$ and E the set of pairs (i, j) for which i divides j . The public is more familiar with social network graphs, in which each person is a vertex and edges exist between pairs of people who are "*friends*".

Since a graph is completely determined by specifying either its adjacency structure or its incidence structure, these specifications provide far more efficient ways of representing a large or complicated graph than a pictorial representation. As computers are more adept at manipulating numbers than at recognising pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. In this chapter, we study various types of matrices associated with a graph.

1.2 Definition and fundamental concepts

1.2.1 Definition

Graph: A graph is a tuple $G = (V, E)$ where V is finite set of vertices and E is finite collection of edges. The set E contain element from the union of the one and two element of V . that is ,each is either a one and two element subset of V .

Conceptually , a graph is formed by vertices and edges connecting the vertices

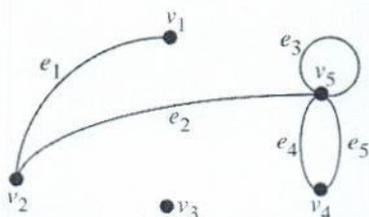
Example.



Formally , a graph is a pair of sets (V, E) , where V is the set of vertices and E is the set of edges , formed by pairs of vertices. E is a multiset , in other words, its elements can occur more than once so that every element has a multiplicity . Often, we label the vertices with letters (for example: a, b, c, . . . or v_1, v_2, \dots) or numbers 1 , 2 , . . . Throughout this lecture material, we will label the elements of V in this way.

Remark. The two edges (u, v) and (v, u) are the same. In other words, the pair is not ordered.

Example.(Continuing from the previous example) We label the edges as follows:



so $E = \{e_1, \dots, e_5\}$

We have the following terminologies:

1. The two vertices u and v are end vertices of the edge (u, v) .
2. Edges that have the same end vertices are parallel .
3. An edge of the form (v, v) is a loop .
4. A graph is simple if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is empty .
6. A graph with no vertices (i.e. V and E are empty) is a null graph .
7. A graph with only one vertex is trivial .
8. Edges are adjacent if they share a common end vertex.
9. Two vertices u and v are adjacent if they are connected by an edge, in

other words, (u, v) is an edge.

10. The degree of the vertex v , written as $d(v)$, is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.

11. A pendant vertex is a vertex whose degree is 1.

12. An edge that has a pendant vertex as an end vertex is a pendant edge.

13. An isolated vertex is a vertex whose degree is 0.

Remark. In this course, we only consider finite graphs, i.e. V and E are finite sets. Since every edge has two end vertices, we get

Theorem 1.2.1. *The graph $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$, satisfies*

$$\sum_{i=1}^n d(v_i) = 2m$$

Proof. Let G be a graph with 'm' number of edges and 'n' number vertices.

$$V = \{v_1, v_2, \dots, v_n\}$$

Degree V_i , $i=1,2,3,\dots,n$ is the number of edges incident to the vertices.

The graph may be a distinct graph on a self loop. Each edge 'e' is either a distinct vertices on a self loop.

If 'e' is a loop on a vertices V . If e incident with two distinct vertices, say v_1, v_2 then e contributes '1' to the degree of each vertex,

Therefore, when we compute the sum,

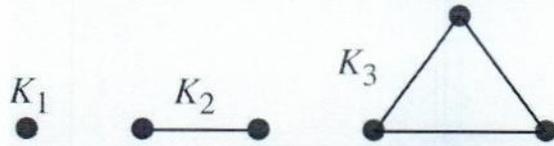
$$\sum_{i=1}^n d(v_i) = d(v_1) + d(v_2) + \dots + d(v_n)$$

$$= 2 \times m.$$

□

Complete graph: A simple graph G is said to be complete graph if every vertex of G is connected with every other vertex of G . It is denoted by K_n with the n of vertices.

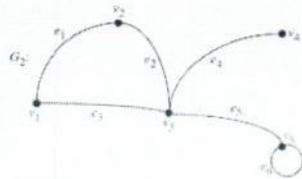
Example.



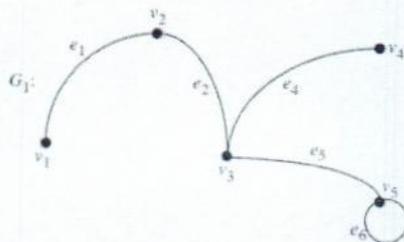
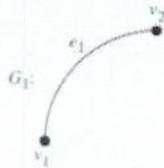
The graph $G_1 = (V_1, E_1)$ is a sub graph of $G_2 = (V_2, E_2)$

1. $V_1 \subseteq V_2$ and
2. Every edge of G_1 is also an edge of G_2 .

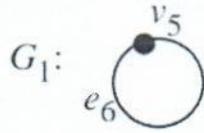
Example. We have the graph



and some of its sub graph are



and



The graph of $G=(V,E)$ induced by the edges set $E_1 \subseteq E$ is:

$$G_1 = (V_1, E_2) =_{def.} \langle E_1 \rangle,$$

where V_1 consists of every end vertex of the edges in E_1

1.2.2 Walks, Trails, Paths,

Remark. There are many different variations of the following terminologies. We will adhere to the definitions given here.

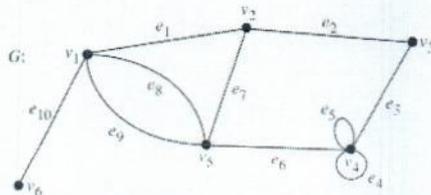
A walk in the graph $G = (V, E)$ is a finite sequence of the form

$$v_{i0}, e_{j1}, v_{i1}, e_{j2}, \dots, e_{jk}, v_{ik},$$

which consists of alternating vertices and edges of G . The walk starts at a vertex. Vertices v_{it}

and v_{it} are end vertices of e_{jt} ($t = 1, \dots, k$). v_{i0} is the initial vertex and v_{ik} is the terminal vertex. k is the length of the walk. A zero length walk is just a single vertex v_{i0} . It is allowed to visit a vertex or go through an edge more than once. A walk is open if $v_{i0} \neq v_{ik}$. Otherwise it is closed.

Example. In the graph



the walk

$$v_2, e_7, v_5, e_8, v_1, e_8, v_5, e_6, v_4, e_5, v_4, e_5, v_4$$

is open. On the other hand, the walk

$$v_4, e_5, v_4, e_3, v_3, e_2, v_2, e_7, v_5, e_6, v_4$$

is closed. A walk is a trail if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v .

Example. (Continuing from the previous example) The walk in the graph

$$v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_4, e_5, v_4, e_4, v_4$$

is a trail.

A trail is a path if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a circuit. For simplicity, we will assume in the future that a circuit is not empty, i.e. its length ≥ 1 . We identify the paths and circuits with the subgraphs induced by their edges.

Example. (Continuing from the previous example) The walk

$$v_2, e_7, v_5, e_6, v_4, e_3, v_3$$

is a path and the walk

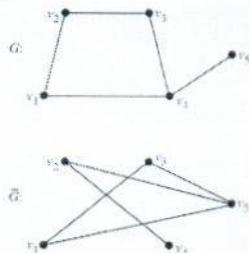
$$v_2, e_7, v_5, e_6, v_4, e_3, v_3, e_2, v_2$$

is a circuit. The walk starting at u and ending at v is called an u - v walk. u and v are connected if there is a u - v walk in the graph (then there is also a u - v path!). If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a u - v walk and a v - w walk, then there is also a u - w walk. A graph is connected if all the vertices are connected to each other. (A trivial graph is connected by convention).

1.2.3 Graph operation

The complement of the simple graph $G = (V, E)$ is the simple graph $\bar{G} = (V, \bar{E})$, where the edges in \bar{E} are exactly the edges not in G .

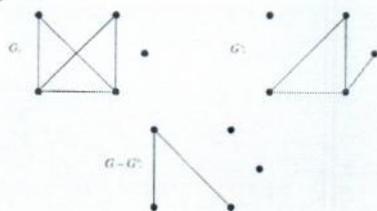
Example.



Example. The complement of the complete graph is the empty graph K_n is the empty graph with n vertices.

Obviously, $\bar{\bar{G}} = G$. If the graphs $G = (V, E)$ and $G' = (V', E')$ are simple and $V' \subseteq V$, then the difference graph is $G - G' = (V, E'')$, where E'' contains those edges from G that are not in G' .

Example.



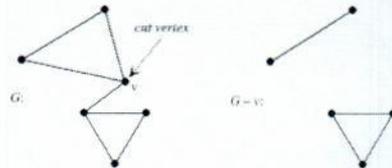
Here are some binary operations between two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$:

- The union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ (simple graph).
- The intersection is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ (simple graph).
- The ring sum $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$ (simple graph).

1.2.4 Cut

A vertex v of a graph G is a cut vertex or an articulation vertex of G if the graph $G - v$ consists of a greater number of components than G .

Example. v is a cut vertex of the graph below:

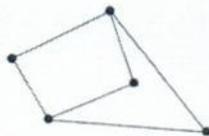


(Note! Generally, the only vertex of a trivial graph is not a cut vertex, neither is an isolated vertex.) A graph is separable if it is not connected or if there exists at least one cut vertex in the graph. Otherwise, the graph is non separable . Example. The graph G in the previous example is separable. Example. The graph below is non separable.

A block of the graph G is a sub graph G_1 of G (not a null graph) such that

- G_1 is non separable, and
- if any other sub graph of G , then $G_1 \cup G_2 = G_1$ or $G_1 \cup G_2$ is separable

Example. The graph below is separable:

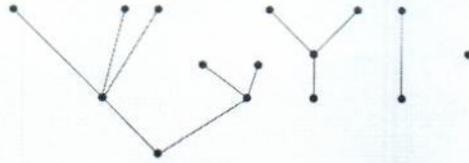


1.2.5 Tree

1.2.5.1 Trees and Forests

A forest is a circuit less graph. A tree is a connected forest. A sub forest is a sub graph of a forest. A connected sub graph of a tree is a sub tree . Generally speaking, a sub forest (respectively sub tree) of a graph is its sub graph, which is also a forest (respectively tree)

Example. Four trees which together form a forest:

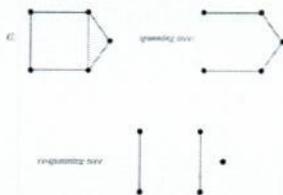


A spanning tree of a connected graph is a sub tree that includes all the vertices of that graph. If T is a spanning tree of the graph G , then

$$G - T =_{\text{def}} T^*$$

is the cospanning tree.

Example.



The edges of a spanning tree are called branches and the edges of the corresponding cospanning tree are called links or chords.

Theorem 1.2.2. *If a tree is not trivial, Then there are at least two pendent vertices.*

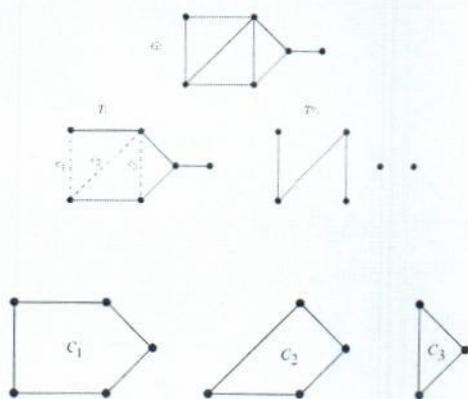
Proof. If a tree has $n (\geq 2)$ vertices, Then the sum of the degree is $2(n-1)$. If every vertex has degree ≥ 2 , then the sum will be $\geq 2n$. On the other hand, if all about one vertex have degree ≥ 2 , then the sum would be $\geq 1 + 2(n-1) = 2n-1$. \square

1.2.5.2 (Fundamental) Circuits and (Fundamental) Cut Sets

If the branches of the spanning tree T of a connected graph G are b_1, \dots, b_{n-1} and the corresponding links of the cospanning tree T^* are c_1, \dots, c_{m-n+1} , then there exists one and only one circuit C_i in $T + c_i$ (which is the sub graph of G induced by the branches of T and c_i) We call this circuit a fundamental circuit. Every spanning tree defines $m - n + 1$ fundamental circuits C_1, \dots, C_{m-n+1} , which together form a fundamental set of circuits. Every fundamental circuit has exactly one link which is not in any other

fundamental circuit in the fundamental set of circuits. Therefore, we can not write any fundamental circuit as a ring sum of other fundamental circuits in the same set. In other words, the fundamental set of circuits is linearly independent under the ring sum operation.

Example.

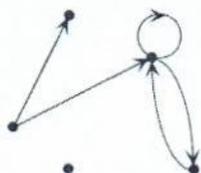


The graph $T-b_i$ has two components T_1 and T_2 . The corresponding vertex sets are V_1 and V_2 . Then, $\langle V_1, V_2 \rangle$ is a cut of G . It is also a cut set of G if we treat it as an edge set because $G-\langle V_1, V_2 \rangle$ has two components. Thus, every branch b_i of T has a corresponding cut set I_i . The cut sets I_1, \dots, I_{n-1} are also known as fundamental cut sets and they form a fundamental set of cut sets. Every fundamental cut set includes exactly one branch of T and every branch of T belongs to exactly one fundamental cut set. Therefore, every spanning tree defines a unique fundamental set of cut sets for G .

1.2.6 Directed graphs

Definition 1.2.3. *Intuitively*, a directed graph or digraph is formed by vertices connected by directed edges or arcs¹.

Example.



Formally, a digraph is a pair (V, E) , where V is the vertices set and E is The set of vertices pair as in "usual" graphs. The difference is that now the elements of E are ordered pairs: the arc from vertex u to vertex v is written as (u, v) and the other pair (v, u) is the opposite direction arc. We also have to keep track of the multiplicity of the arc (direction of a loop is irrelevant). We can pretty much use the same notions and results for digraphs from Chapter 1. However:

1. Vertex u is the initial vertex and vertex v is the terminal vertex of the arc (u, v) . We also say that the arc is incident out of u and incident into v .
2. The out-degree of the vertex v is the number of arcs out of it (denoted $d^+(v)$) and the in-degree of v is the number of arcs going into it (denoted $d^-(v)$).
3. In the directed walk (trail, path or circuit).

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \dots, e_{j_k}, v_{i_k}$$

v_{i_i} is the initial vertex and $v_{i_{i-1}}$ is the terminal vertex of the arc.

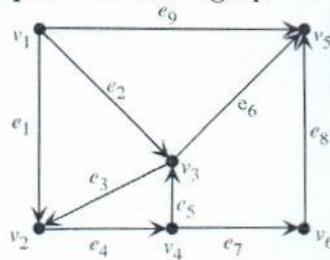
4. When we treat the graph (V, E) as a usual undirected graph, it is the underlying undirected graph of the digraph $G=(V, E)$, denoted G_u .
5. Digraph G is connected if G_u is connected. The components of G are the directed sub- graphs of G that correspond to the components of G_u . The vertices of G are connected if they are connected in G_u . Other notions for undirected graphs can be used for digraphs as well by dealing with the underlying undirected graph.
6. Vertices u and v are strongly connected if there is a directed u - v path and also a directed v - u path in G .
7. Digraph G is strongly connected if every pair of vertices is strongly

connected. By convention, the trivial graph is strongly connected.

8. A strongly connected component H of the digraph G is a directed sub graph of G (not a null graph) such that H is strongly connected, but if we add any vertices or arcs to it, then it is not strongly connected anymore.

Every vertex of the digraph G belongs to one strongly connected component of G (compare to Theorem 1.3). However, an arc does not necessarily belong to any strongly connected component of G .

Example. For the digraph G



the strongly connected components are $(\{v_1\})$, $(\{v_2, v_3, v_4\})$, $(\{v_5\})$, $(\{v_6\})$ and $(\{v_5, v_6\})$.

The condensed graph G_c of the digraph G is obtained by contracting all the arcs in every strongly connected component.

1.2.7 Directed tree

A directed graph is quasi-strongly connected if one of the following conditions holds for every pair of vertices u and v :

- (i) $u=v$ or
- (ii) there is a directed $u-v$ path in the digraph or
- (iii) there is a directed $v-u$ path in the digraph or
- (iv) there is a vertex w so that there is a directed $w-u$ path and a directed $w-v$ path.

Example. (Continuing from the previous example) The digraph G is quasi-strongly connected

Quasi-strongly connected digraphs are connected but not necessarily strongly connected. The vertex v of the digraph G is a root if there is a directed path from v to every other vertex of G .

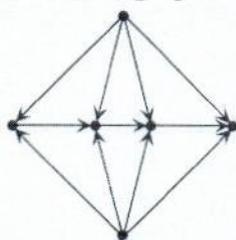
Example. (Continuing from the previous example) The digraph G only has

one root, v_1 .

1.2.8 Acyclic Directed Graphs

A directed graph with at least one directed circuit is said to be cyclic . A directed graph is acyclic otherwise. Obviously, directed trees are acyclic but the re verse implication is not true.

Example. The digraph



is acyclic but it is not a directed tree.

Chapter 2

MATRICES REPRESENTATION OF GRAPHS

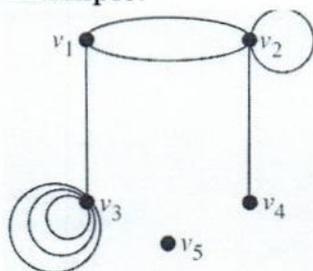
2.1 Definition

Adjacency Matrix: The adjacency matrix of the graph $G = (V, E)$ is an $n \times n$ matrix $D = (d_{ij})$, where n is the number of vertices in G , $V = \{v_1, \dots, v_n\}$ and.

d_{ij} = number of edges between v_i and v_j .

In particular, $d_{ij} = 0$ if (v_i, v_j) is not an edge in G . The matrix D is symmetric, i.e. $D^T = D$.

Example.

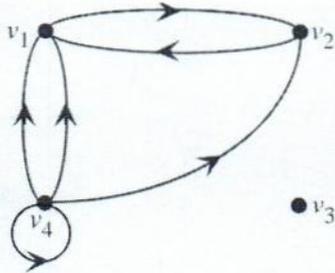


$$D = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Obviously, an adjacency matrix defines a graph completely up to an isomorphism. The adjacency matrix of a directed graph G is $D = (d_{ij})$, where

d_{ij} = number of arcs that come out of vertex v_i and go into vertex v_j .

Example.



$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

The all-vertex incidence matrix of a non-empty and loop less graph $G = (V, E)$ is an $n \times m$ matrix $A = (a_{ij})$, where n is the number of vertices in G , m is the number of edges in G and

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is an end vertex of } e_j \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Theorem 2.1.1. *The determinant of an incidence matrix of a nontrivial tree is ± 1 , regardless of whether the tree is a directed graph or not.*

Proof. We use induction on n , the number of vertices in the tree.

Induction Basis: : $n=2$ and it is obvious.

Induction Hypothesis: : The theorem is true for $n \leq k$. ($k \geq 2$)

Induction Statement: : The theorem is true for $n=k+1$.

Induction Statement Proof: : Let T be a tree which has $k+1$ vertices and let A be an (arbitrary) incidence matrix of T . T has at least two pendant vertices (Theorem 2.3). We choose a pendant vertex v_i which is not the reference vertex of A and the edge e_t which is incident on v_i . Then,

$$a_{it} = (\pm)1 \text{ and } a_{ij} = 0, \text{ when } j \neq t.$$

We expand the determinant of $|A|$ by the i^{th} row:

$$|A| = (\pm)(-1)^{i+t} |A'|,$$

where A' is the minor corresponding to a_{it} . We write $T' = T - v_i$ which is

also a tree (v_i is a pendant vertex). We use the induction hypothesis to get $|A'| = \pm 1$ because A' is obviously an incidence matrix of T' .

□

Corollary 2.1.2. *If the digraph G has no loops, then the rank of its all-vertex incidence matrix is $\rho(G)$.*

Proof. If we rearrange the rows or columns of the all-vertex incidence matrix, the rank of the matrix will not change. Let us rearrange the vertices and arcs to group them by components. Then, the all-vertex incidence matrix is a block diagonal matrix in which each block is an all-vertex incidence matrix

of a component.
$$\begin{pmatrix} 1^{st} \text{ component} & & & 0 \\ & 2^{nd} \text{ component} & & \\ & & \ddots & \\ 0 & & & k^{th} \text{ component} \end{pmatrix}$$

We denote n_i as the number of vertices in the i^{th} component. Every component has a spanning tree whose incidence matrix has a nonzero determinant by Theorem 3.1, i.e. the matrix is not singular. The all-vertex incidence matrix of the i^{th} component is obtained by adding columns and one row to an incidence matrix of the corresponding spanning tree. The row added is linearly dependent of other rows so that the rank of this matrix is the same as the rank of the incidence matrix ($=n_i - 1$). Notice that in the special case when a component is trivial, the rank is zero $= 1 - 1$. Therefore,

rank of $A = \text{sum of the ranks of the components}$

$$= (n_1 - 1) + \dots + (n_k - 1)$$

$$= n_1 + \dots + n_k - k = \rho(G)$$

□

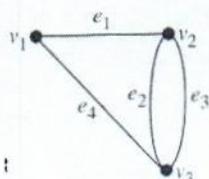
Remark. From this proof, we can also get a basis for the row space and the column space of the all-vertex incidence matrix. The columns corresponding to the branches of the spanning forest of G are a basis of the column space. We can get a basis of the row space by removing one row out of each component block.

2.2 Cut Matrix

If all the cuts of a nontrivial and loop less graph $G = (V, E)$ are I_1, I_2, \dots, I_t , then the cut matrix of G is a $t \times m$ matrix $Q = (q_{ij})$, where m is the number of edges in G and

$$q_{ij} = \begin{cases} 1 & \text{if } e_j \in I_i \text{ (the cut is interpreted as an edge set)} \\ 0 & \text{other wise.} \end{cases} \quad (2.2)$$

Example. For the graph



the cuts are $I_1 = \{e_1, e_4\}$, $I_2 = \{e_2, e_3, e_4\}$ and $I_3 = \{e_1, e_2, e_3\}$. The cut matrix is

$$Q = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{matrix} I_1 \\ I_2 \\ I_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Remark. If the graph has n vertices, then it has $\frac{1}{2}(2^n - 2) = 2^{n-1} - 1$ cuts. Usually, there are not this many distinct edge sets. For the cut matrix, we only take one cut corresponding to an edge set so that there would not be repeated rows. Even so, there are usually too many rows.

If G is a nontrivial and loopless digraph, then we assign an arbitrary direction to every cut $\langle V_1, V_2 \rangle$: the orientation of $\langle V_1, V_2 \rangle$ is from V_1 to V_2 . In other words, we consider oriented cuts and we pick only one direction from the two possibilities. then, the cut matrix $Q = (q_{ij})$ is

$$q_{ij} = \begin{cases} 1 & \text{if } e_j \in I_i \text{ and they are in the same direction} \\ -1 & \text{if } e_j \in I_i \text{ and they are in the opposite direction} \\ 0 & \text{other wise.} \end{cases} \quad (2.3)$$

Theorem 2.2.1. Every row of the cut matrix of a digraph can be expressed

in two different ways as a linear combination of the rows of the all-vertex incidence matrix. The non-zero coefficients are either all = +1 or all = -1.

Proof. Let Q be the cut matrix of a digraph $G = (V, E)$ and let A be the all-vertex incidence matrix. Let $\langle V_1, V_2 \rangle$ (note that it is oriented) be the cut corresponding to the i th row of Q . Reindexing if needed, we can assume that

$$V_1 = \{v_1, \dots, v_r\} \text{ and } V_2 = \{v_{r+1}, \dots, v_n\}.$$

We write

$$q_i = i^{\text{th}} \text{ row of } Q \text{ and } a_t = t^{\text{th}} \text{ row of } A$$

We show that

$$q_i = \sum_{t=1}^r a_t = - \sum_{t=r+1}^n a_t,$$

which proves the theorem. Let $(v_p, v_q) = e_k$ be the k^{th} of G . Then,

$$a_{pk} = k^{\text{th}} \text{ element of the vector } a_p = 1$$

$$a_{qk} = k^{\text{th}} \text{ element of the vector } a_q = -1$$

We get four cases:

- $v_p \in V_1$ and $v_q \in V_2$: Now $p \leq r$ and $q \geq r+1$ so $q_{ik} = 1$ and

$$q_{ik} = \sum_{t=1}^r a_{tk} = - \sum_{t=r+1}^n a_{tk}.$$

- $v_p \in V_2$ and $v_q \in V_1$: Now $p \geq r+1$ and $q \leq r$ so $q_{ik} = -1$ and

$$q_{ik} = \sum_{t=1}^r a_{tk} = - \sum_{t=r+1}^n a_{tk}.$$

- $v_p \in V_1$ and $v_q \in V_1$: Now $p \leq r$ and $q \leq r$ so $q_{ik} = 0$ and

$$q_{ik} = \sum_{t=1}^r a_{tk} = -a_{r+1,k} - \dots - a_{nk}.$$

- $v_p \in V_2$ and $v_q \in V_2$: Now $p \geq r+1$ and $q \geq r$ so $q_{ik} = 0$ and

$$q_{ik} = a_{1k} + \dots + a_{rk} = - \sum_{t=r+1}^n a_{tk}.$$

The statements above are valid for every k .

□

Corollary 2.2.2. *The rank of the cut matrix of a digraph G is $\rho(G)$.*

Proof. The all-vertex incidence matrix A of G is also a sub matrix of the cut matrix Q of G . Then, (by Corollary of Theorem 3.1)

$$\text{rank}(Q) \geq \text{rank}(A) = \rho(G)$$

On the other hand, by Theorem 4.2, every row of Q can be expressed as a linear combination of the rows of A . Therefore,

$$\text{rank}(Q) = \text{rank}(A) = \rho(G)$$

□

Another consequence is that the cut matrix Q can be expressed as

$$Q = A_1 A,$$

where the elements of A_1 are 0 or ± 1 . In addition, the matrix A_1

If the graph G is connected, then it has a spanning tree T and an associated fundamental cut set. The fundamental cut sets are also cuts (when cuts are interpreted as edge sets). Therefore, the cut matrix Q of G has a submatrix Q_f that corresponds to these fundamental cut sets. This matrix is called the fundamental cut set matrix. Similarly, the connected digraph G has a fundamental cut set matrix: if we interpret a fundamental cut set as a set, then the direction of the cut is chosen to be the same as the direction of the corresponding branch of T . If we rearrange the edges of G so that we have the branches first and sort the fundamental cut sets in the same order, then we get the fundamental cut set matrix in the form

$$Q_f = (I_{n-1} \mid Q_{fc}),$$

where I_{n-1} is the identity matrix with $n - 1$ rows. The rank of Q_f is thus $n - 1 = \rho(G)$.

2.3 Circuit Matrix

We consider a loop less graph $G = (V, E)$ which contains circuits. We enumerate the circuits of $G : C_1, \dots, C_l$. The circuit matrix of G is an $l \times m$ matrix $B = (b_{ij})$ where

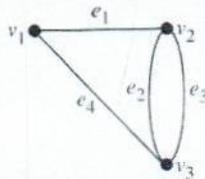
$$q_{ij} = \begin{cases} 1 & \text{if the arc } e_j \text{ is in the circuit } C_i \\ 0 & \text{other wise.} \end{cases} \quad (2.4)$$

(as usual, $E = \{e_1, \dots, e_m\}$).

The circuits in the digraph G are oriented , i.e. every circuit is given an arbitrary direction for the sake of defining the circuit matrix. After choosing the orientations, the circuit matrix of G is $B = (b_{ij})$ where

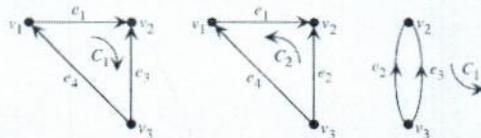
$$q_{ij} = \begin{cases} 1 & \text{if the arc } e_j \text{ is in the circuit } C_i \text{ and they in the same direction} \\ -1 & \text{if the arc } e_j \text{ is in the circuit } C_i \text{ and they in the opposite direction} \\ 0 & \text{other wise.} \end{cases} \quad (2.5)$$

Example. For the directed graph



the circuits are

and the circuit matrix is



and the circuit matrix is

$$B = \begin{matrix} & e_1 & e_2 & e_3 & e_4 \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \end{matrix}$$

If the graph G is connected and contains at least one circuit, then it has a c ospanning tree T^* and the corresponding fundamental circuits. By choosing th e corresponding rows of the circuit matrix B , we get an $(m - n + 1) \times m$ matrix B_f , called the fundamental circuit matrix . Similarly,

a connected digraph G with at least one circuit has a fundamental circuit matrix: the direction of a fundamental circuit is the same as the direction of the corresponding link in T^* .

When we rearrange the edges of G so that the links of T^* come last and sort the fundamental circuits in the same order, the fundamental circuit matrix takes the form

$$B_f = (B_{ft} \mid I_{m-n-1}),$$

Where I_{m-n-1} is the identity matrix with $m - n + 1$ rows. The rank of B_f is thus $m - n + 1 = \mu(G)$ and the rank of B is $\geq m - n + 1$.

Theorem 2.3.1. *If the digraph G contains at least one circuit, then the rank of its circuit matrix B is $\mu(G)$. Furthermore, if G is connected, then the circuit matrix B can be expressed as $B = B_2 B_f$, where the matrix B_2 consists of 0's and 1's and the cut matrix Q can be expressed as $Q = Q_1 Q_f$, where the matrix Q_1 consists of 0's and ± 1 's.*

Proof. First we consider the case when G is connected. We choose a spanning tree T of G and rearrange the m edges of G so that the branches of T come first and the links of T^* come last. We sort the fundamental cut sets in the same order as the branches and links. Then,

$$Q_f = (I_{n-1} \mid Q_{fc}) \text{ and } B_f = (B_{ft} \mid I_{m-n-1}),$$

The blocks of B can be constructed in a similar way:

$$B = (B_1 \mid B_2).$$

Since Q_f is a sub matrix of Q and B_f is a sub matrix of B ,

$$\begin{aligned} O &= B_f Q_f^T = (B_{ft} \mid I_{m-n-1}) (I_{n-1} \mid Q_{fc})^T = (B_{ft} \mid I_{m-n-1}) \begin{pmatrix} I_{n-1} \\ Q_{fc}^T \end{pmatrix} \\ &= B_{ft} I_{n-1} + I_{m-n-1} Q_{fc}^T = B_{ft} = Q_{fc}^T. \end{aligned}$$

Hence

$$B_{ft} = -Q_{fc}^T.$$

Furthermore, since Q_f is a submatrix of Q , we can use the same theorem to get

$$O = B Q_f^T = (B_1 \mid B_2) (I_{n-1} \mid Q_{fc})^T = (B_1 \mid B_2) \begin{pmatrix} I_{n-1} \\ Q_{fc}^T \end{pmatrix}$$

$$= B_1 I_{n-1} + B_2 Q_{fc}^T = B_1 - B_2 B_{ft}.$$

Hence

$$B = (B_2 B_{ft} \mid B_2) = B_2 (B_{ft} \mid I_{m-n-1}) = B_2 B_f,$$

as claimed. In the same way, Q can be expressed as $Q = Q_1 Q_f$, as claimed, which is clear anyway since the rank of Q is $n - 1$ and its elements are 0's and 1's.

Every row of B is a linear combination of the rows corresponding to the fundamental circuits and the rank of B is at most equal to the rank of $B_f = m - n + 1$. On the other hand, as we pointed out earlier, the rank of B is $\mu m - n + 1$. Thus, $\text{rank}(B) = m - n + 1 (= \mu(G))$ for a connected digraph.

In the case of a disconnected digraph G (which contains at least one circuit), it is divided into components ($k \geq 2$ components) and the circuit matrix B is divided into blocks corresponding to the components (compare to the proof of the corollary of Theorem 3.1), in which case

$$\text{rank}(B) = \sum_{i=1}^k (m_i - n_i + 1) = m - n + k = \mu(G).$$

□

2.4 An Application: Stationary Linear Networks

A stationary linear network is a directed graph G that satisfies the following conditions:

1. G is connected.
2. Every arc of G belongs to some circuit and there are no loops in G .
3. Every arc e_j in G is associated with a number i_j called the through-quantity or flow. If there are m arcs in G , then we write

$$i = \begin{pmatrix} i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_m \end{pmatrix}$$

(through-vector).

4. Every vertex v_i in G is associated with a number p_i called the potential . Furthermore, the across-quantity or potential difference of the arc $e_j = (v_{i1}, v_{i2})$ is

$$u_j = p_{i2} - p_{i1}.$$

If there are n vertices and m arcs in G , then we write

$$p = \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{pmatrix}$$

and

$$u = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{pmatrix}$$

(potential vector and across-vector). (Potentials are rarely needed.)

5. Every arc e_j is one of the following:

- (a) a *component*¹ , for which there is an associated number r_j . r_j is constant ($\neq 0$) (stationarity) and the following equation links the quantities:

$$u_j = i_j r_j \text{ (linearity).}$$

- (b) a through-source , for which the through-quantity i_j is fixed.

- (c) an across-source , for which the across-quantity u_j is fixed.

6. (Kirchhoff's Through-Quantity Law) The sum of the through-quantities of an oriented cut of G is zero when the cut is interpreted as an edge set and the sign of a through-quantity is changed if the directions of a cut and an arc are different.

7. (Kirchhoff's Across-Quantity Law) The sum of the across-quantities of an oriented circuit of G is zero when the sign of an across-quantity is changed if the directions of a circuit and an arc are different

Example. A typical stationary linear network is an electrical circuit with linear resistors, constant current sources and constant voltage sources. The components are resistors and r_j are the resistances. Equation 5.(a) is *Ohms Law*.

We take a spanning tree T of a stationary linear network G , its fundamental cut matrix Q_f and its fundamental circuit matrix B_f . Let us rearrange the arcs in these matrices and vectors i and u like we did before. That is, the branches of T will come first followed by the links of T^* . *Kirchhoffs Laws* can then be written as

$$Q_i=0 \text{ and } B_u=0.$$

On the other hand, the rows of the fundamental cut matrix Q_f span all the rows of Q , and similarly rows of the fundamental circuit matrix B_f span the rows of B . Then, *Kirchhoffs Laws* can also be written as

$$Q_f i = o_{n-1} \text{ and } B_f u = o_{m-n+1}.$$

Let us form the diagonal matrices $K=[k_1, \dots, k_m]$ and $L = [l_1, \dots, l_m]$, where

$$k_j = \begin{cases} -r_j & \text{if } e_j \text{ is a component} \\ -1 & \text{if } e_j \text{ is a through-source} \\ 0 & \text{if } e_j \text{ is a across-source} \end{cases} \quad (2.6)$$

and

$$l_j = \begin{cases} 1 & \text{if } e_j \text{ is a component} \\ 0 & \text{if } e_j \text{ is a through-source} \\ 1 & \text{if } e_j \text{ is a across-source} \end{cases} \quad (2.7)$$

and the m -vector $s=(s_1, \dots, s_m)^T$, where

$$s_j = \begin{cases} 0 & \text{if } e_j \text{ is a component} \\ i_j & \text{if } e_j \text{ is a through-source} \\ u_j & \text{if } e_j \text{ is a across-source} \end{cases} \quad (2.8)$$

Then, all the information can be expressed as a system of linear equations

$$\left(\begin{array}{c|c} K & L \\ \hline Q_f & O_{n-1} \times m \\ \hline O_{m-n+1} \times m & B_f \end{array} \right) \begin{pmatrix} i \\ u \end{pmatrix} = \begin{pmatrix} S \\ O_{n-1} \\ O_{m-n+1} \end{pmatrix}$$

Remark. The same procedure can be applied to form state (differential) equations for dynamic networks, which have non stationary components.

The matrix of this system of linear equations does not have to be non singular and the system does not even have to have a unique solution at all. For example, in the matrix above, we can easily see that it is singular if some circuit only consists of across-sources or if some cut only consists of through-sources. As a matter of fact, this is the only case when the through and across quantities are not defined uniquely if the constants r_j are real numbers with the same sign (and often otherwise too). We choose a specific spanning tree T to explore these concepts more carefully:

Lemma. If no cut of G consists of only through-sources and no circuit of G consists of only across-sources, then G has a spanning tree T such that every across-source is a branch of T and every through-source is a link of T^* .

Proof. If G satisfies the hypothesis, then we first choose a digraph M which has every vertex and across-source (arc) of G . There are no circuits in this digraph. Then we add components to M one by one and try to come up with a spanning tree. If this fails at some point, then G has a cut with only through-sources, which is impossible.

2.5 Matrices over GF(2) and Vector Spaces of Graphs

The set $\{0, 1\}$ is called a field (i.e. it follows the same arithmetic rules as real numbers) if addition and multiplication are defined as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

In this case $1 = 1$ and $1^{-1} = 1$. This is the field GF(2).

If we think of the elements 0 and 1 of the all-vertex incidence, cut,

fundamental cut, circuit and fundamental circuit matrices of a ("undirected") graph as elements of the field $GF(2)$, then Theorems 4.1, 4.2, 4.4, 4.5 and their corollaries also apply to "undirected graphs". (Keep in mind that $1 = 1$ in the field $GF(2)$.) The proofs are the same

For "undirected" graphs, the vector spaces are over the field $GF(2)$. For directed graphs, the vector spaces are real (i.e. over the field R). The row space of the cut matrix of a (di)graph is the cut space. Similarly, the row space of the circuit matrix is the circuit space. The dimension of the cut space is the rank of the (di)graph and the dimension of the circuit space is the nullity of the (di)graph. Furthermore, the cut space and the circuit space are orthogonal complements. (All of these statements follow directly from the results above).

Often, we deal with the above mentioned spaces through sub graphs, i.e. we identify a vector with the subgraph generated by the corresponding arcs. In the case of undirected graphs, the addition of $GF(2)$ vector corresponding to the ring sum operation.

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Signature of the Students by whom Project "A Study on Matrices & Vector Spaces of Graphs" is carried out.

Sl No	Roll Number	Name of the Student	Signature
1	BS17-111	SUDHANSU SEKHAR ROUT	Sudhansu Sekhar Rout
2	BS17-119	MANASI MOHANTY	Manasi Mohanty
3	BS17-132	DIBYASUNDAR BEHERA	Dibyasundar Behera
4	BS17-135	ANSUMAN SAHOO	Ansuman Sahoo